Structural Parameterizations of b-Coloring*

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Abstract

The b-Coloring problem, which given a graph G and an integer k asks whether G has a proper k-coloring such that each color class has a vertex adjacent to all color classes except its own, is known to be FPT parameterized by the vertex cover number and XP and W[1]-hard parameterized by clique-width. Its complexity when parameterized by the treewidth of the input graph remained an open problem. We settle this question by showing that b-Coloring is XNLP-complete when parameterized by the pathwidth of the input graph. Besides determining the precise parameterized complexity of this problem, this implies that b-Coloring parameterized by pathwidth is W[t]-hard for all t, and resolves the parameterized complexity of b-Coloring parameterized by treewidth. We complement this result by showing that b-Coloring is FPT when parameterized by neighborhood diversity and by twin cover, two parameters that generalize vertex cover to more dense graphs, but are incomparable to pathwidth.

1 Introduction

A *b-coloring* of a graph G is a proper vertex-coloring such that each color class has a vertex, called *b-vertex*, that has a neighbor in each color class except its own. This problem originated in the study of the color-suppressing heuristic for the Graph Coloring problem: Start with any proper coloring of G, and keep on *suppressing* color classes as long as you can. Here, a color class C can be suppressed, if for each vertex with color C, there is a color $C' \neq C$ that does not yet appear in its neighborhood. This allows us to recolor all vertices in C and thereby lower the number of colors by one. The colorings which do not allow for further improvements are exactly the b-colorings, so the largest integer k such that a graph G admits a b-coloring with k colors determines the worst-case behavior of this heuristic, when applied to G. This quantity is referred to as the *b-chromatic number*. In this work, we study the following decision problem related to b-colorings. For more details on computational problems associated with b-colorings, we refer to [21, 22, 27].

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b-Coloring

Input: Graph G, integer k
Question: Does G have a b-coloring with k colors?

This problem is known to be NP-complete, even when the number of colors is fixed [21]. The complexity of b-Coloring has been studied on several graph classes, see for instance [8, 9, 10, 11, 19, 21, 25, 28]. Recent work also considered first *structural parameterizations*: Jaffke, Lima, and Lokshtanov [22] showed that b-Coloring is FPT when parameterized by the vertex cover number of the input graph, and XP and W[1]-hard when parameterized by clique-width. Arguably the most prominent parameter (between the vertex cover number and the clique-width) is *treewidth*. Parameterized by treewidth plus the number of colors, b-Coloring is FPT [1, 22]. However, in the setting when the number of colors is part of the input, the parameterized complexity of b-Coloring by treewidth remained open. In fact, this problem has already been stated explicitly over a decade ago, see for instance [27]. Observe that even n-vertex forests can have b-colorings with $\sqrt{n}$ colors (think of a forest where each of the $\sqrt{n}$ components is a star with $\sqrt{n}-1$ leaves).

The first main result of this work is to resolve this open problem, showing hardness. We prove a stronger hardness result than W[1]-hardness by treewidth, namely XNLP-completeness by the more restrictive parameter pathwidth. The class XNLP has recently been coined by Bodlaender et al. [6], derived from earlier work of Elberfeld, Stockhusen, and Tantau [14], as a means of addressing the question of *completeness* of parameterized problems. XNLP is the class of parameterized problems that can be solved by a nondeterministic algorithm using $f(k) \cdot n^c$ time and $f(k) \cdot \log n$ space, where k is the parameter, n the input size, c a constant, and f a computable function. Several parameterized problems have been shown to be XNLP-complete [4, 5, 6], most relevant for our work problems parameterized by linear width measures [4, 5]. While XNLP-hardness reductions are often very similar to reductions proving W-hardness, they yield a much stronger result. As XNLP contains the entire W-hierarchy [6], XNLP-hardness implies W[t]-hardness for all $t \in \mathbb{N}$.

Theorem 1. *b-Coloring parameterized by pathwidth is XNLP-complete, and therefore W[t]-hard for all $t \in \mathbb{N}$.*

Notice that the previous theorem implies that b-Coloring parameterized by treewidth or by clique-width is W[t]-hard for all $t \in \mathbb{N}$, therefore resolving the open question of the parameterized complexity of b-Coloring parameterized by treewidth [22, 27], and significantly strengthening the W[1]-hardness result by clique-width [22].

We complement Theorem 1 with two positive results about generalizations of the vertex cover number. The FPT-algorithm parameterized by vertex cover of [22] essentially follows from two observations. First, that the number of colors in any b-coloring of a graph with vertex cover number t is bounded by a function of t. Second, that the treewidth is always at most the vertex cover number. Therefore, the algorithm follows by an FPT-algorithm parameterized by treewidth plus number of colors. The generalizations we consider here, neighborhood diversity and twin-cover, both extend the vertex cover number to simply structured dense graphs; in particular, complete graphs have twin-cover number 0 and neighborhood diversity 1. This means that in both parameterizations, the number of colors in a b-coloring can be as high as $\Omega(n)$. Nevertheless, we obtain FPT-algorithms in both cases.

Theorem 2. *b-Coloring parameterized by the twin-cover number or by the neighborhood diversity of a graph is fixed-parameter tractable.*

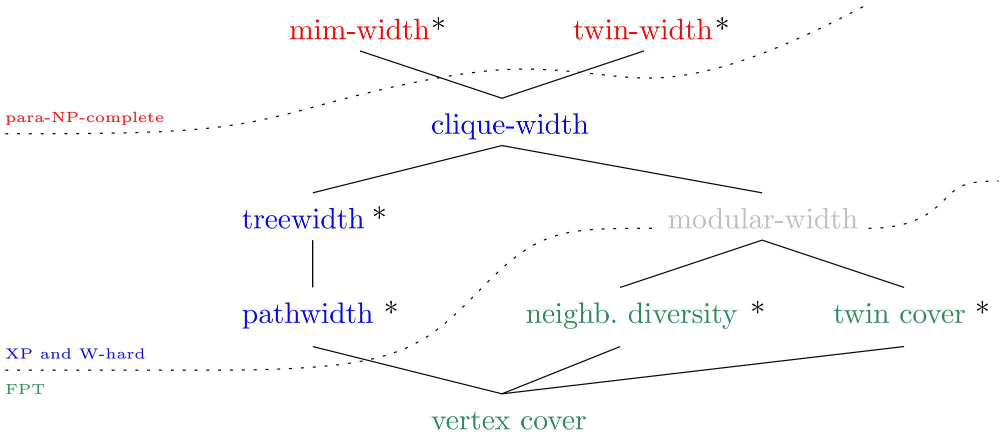

Figure 1: Known results about structural parameterizations of b-Coloring. Results marked with * can be found in this work. The complexity of b-Coloring parameterized by modular-width remains open.

Lastly, we observe by two trivial reductions that the XP-algorithms parameterized by clique-width cannot be extended to the more general width measures mim-width and twin-width. In both cases, this follows directly from known hardness results of Graph Coloring on certain graph classes. In the case of twin-width, this even holds when the number of colors is a fixed constant. This stronger hardness result does not hold for mim-width, as b-Coloring is expressible in DN logic by a sentence whose length depends only on the number of colors, and therefore XP parameterized by the mim-width of a given decomposition plus the number of colors [3].

Observation 3. b-Coloring is NP-complete on graphs of linear mim-width 2, and on graphs of twin-width at most 8. In the case of twin-width, para-NP-hardness even holds when the number of colors is any fixed constant $q \geq 3$.

We summarize these results in Figure 1. Note that the parameterization modular-width, which is a common generalization of neighborhood diversity and twin-cover, remains open. The remainder of the paper is organized as follows. In Section 2, we give the necessary background and definitions, and justify Observation 3. In Section 3, we consider the parameterization by pathwidth and prove Theorem 1, and in Sections 4 and 5 we consider neighborhood diversity and twin-cover, respectively, to prove Theorem 2. We conclude in Section 6.

2 Preliminaries

Basic notations and definitions. For two integers $a \leq b$ we let $[a..b] = \{a, a+1, \ldots, b\}$, and for a positive integer a, we let $[a] = [1..a]$. All graphs considered here are finite and simple. For an (undirected or directed) graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. For an edge $\{u, v\} \in E(G)$, we use the shorthand "uv". If G is a directed graph, then denoting the edge $e = (u, v) \in E(G)$ by uv also points to e being directed from u to v. Given an undirected graph G, an *orientation* of G, denoted by $\overrightarrow{G}$, is a directed graph obtained from G by replacing each edge $\{u, v\} \in E(G)$ by either (u, v) or (v, u). The *neighborhood of a vertex* v is defined as $N(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *closed neighborhood of* v is defined as $N[v] = N(v) \cup \{v\}$. The *neighborhood of a set* $S \subseteq V(G)$ is defined similarly, that is, $N(S) = \{u \in V(G) \setminus S \mid us \in E(G) \text{ for some } s \in S\}$. The *closed neighborhood of* S is $N[S] = N(S) \cup S$. A set $S \subseteq V(G)$ is *independent* if for all pairs of distinct $u, v \in S$, $uv \notin E(G)$. A set $C \subseteq V(G)$ is a *clique* if for all pairs of distinct $u, v \in C$, $uv \in E(G)$. A *star* is an undirected graph with one special vertex called the *center* that is adjacent to all of the remaining vertices, called *leaves*, which form an independent set.

Colorings. A *proper coloring with k colors* of a graph G is a partition of $V(G)$ into k independent sets, called *color classes*. For a graph G and a proper coloring of G, a vertex $v \in V(G)$ is a *b-vertex* if it has a neighbor in all color classes except its own. A *b-coloring with k colors* of a graph G is a proper coloring of G such that each color class contains a b-vertex. We call such a coloring a *k-b-coloring*. In this work we consider the following problem.

b-Coloring

Input:	Graph G, integer k
Question:	Does G have a b-coloring with k colors?

2.1 Width measures

We now define the width measures relevant for this work and state some known facts about them for completeness.

Definition 4 (Module, Modular Partition, Quotient Graph). A *module* of a graph G is a set of vertices $M \subseteq V(G)$ such that for all $v \in V(G) \setminus M$, either $M \subseteq N_G(v)$ or $M \cap N_G(v) = \emptyset$. A partition $\mathcal{P}$ of $V(G)$ is a *modular partition* if all parts of $\mathcal{P}$ are modules in G. The *quotient graph* of $\mathcal{P}$, denoted by $G/\mathcal{P}$ is the graph whose vertex set is $\mathcal{P}$ such that for all $P, Q \in \mathcal{P}$, $PQ \in E(G/\mathcal{P})$ if P is complete to Q in G and $PQ \notin E(G/\mathcal{P})$ if P is anti-complete to Q in G.

Definition 5 (Neighborhood Diversity [26]). An *ND-partition* of a graph G is a modular partition $\mathcal{P}$ of $V(G)$ such that each part of $\mathcal{P}$ is either a clique (called a *clique part*) or an independent set (called *independent part*) in G. The *neighborhood diversity* of a graph G is the minimum number of parts in any ND-partition of G.

Remark 6. Note that the neighborhood diversity can also be defined as follows: for a graph G, say that two vertices u, v are equivalent if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Each equivalence class consists of a complete part and an independent part which gives the optimal ND-partition in polynomial time.

Definition 7 (Twin Cover [16]). A set $S \subseteq V(G)$ of a graph G is a *twin cover*, if for each edge $uv \in V(G)$, either (i) $\{u, v\} \cap S \neq \emptyset$, or (ii) or u and v are twins in G. The *twin cover number* of G is the smallest size of any twin cover of G.

Observation 8. If S is a twin-cover of a graph G then each connected component of $G - S$ is a clique consisting of twins.

Let G be a graph and S a vertex cover of size k. It is clear that S is also a twin cover. Moreover, for each $A \subseteq S$, let $P_A \subseteq V(G) \setminus S$ be the set of all vertices v with $N(v) = A$. Then, the partition of $V(G)$ consisting of all singletons of S plus the sets P_A for all $A \subseteq S$ is an ND-partition of G, so G has neighborhood diversity at most $k + 2^k$ [26].

Neighborhood diversity and twin cover number are incomparable: consider for instance $K_{n,n}$, a complete bipartite graph with n vertices on each side. The neighborhood diversity of $K_{n,n}$ is two, as the natural bipartition of its vertices is an ND-partition. On the other hand, each twin cover of $K_{n,n}$ has size at least n (it has to fully contain one of the sides).

Conversely, consider the windmill graph W_n with n petals, that is, a collection of n triangles where each triangle has one special vertex that is identified with all other special vertices. The Twin cover number of W_n is one (just take the vertex resulting from identifying all the special vertices), while the neighborhood diversity of W_n is $n+1$. For the lower bound, observe that no two non-special vertices from distinct triangles can be in the same part of an ND-partition.

Definition 9. Let G be a graph. A *path decomposition* of G is a sequence $\mathcal{B} = B_1, \ldots, B_d$ of subsets of $V(G)$ called *bags* covering $V(G)$ such that:

(i) For each edge $e \in E(G)$, there is some $i \in [d]$ such that $e \subseteq B_i$.

(ii) For each $h, i, j \in [d]$ with $h < i < j$, $B_h \cap B_j \subseteq B_i$.

The *width* of $\mathcal{B}$ is $\max_{i \in [d]} |B_i| - 1$, and the *pathwidth* of G is the smallest width of all its path decompositions.

Membership in XNLP of b-Coloring parameterized by pathwidth will follow from the membership of b-Coloring parameterized by a linear width measure with more expressive power than pathwidth, namely one that is equivalent to linear clique-width. We define it next and show its relation to pathwidth.

Definition 10. Let G be a graph and $S \subseteq V(G)$. The *module number* of S is the number of equivalence classes of the equivalence relation $\sim_S$ defined as: $u \sim_S v \Leftrightarrow N(u) \cap (V(G) \setminus S) = N(v) \cap (V(G) \setminus S)$. Let $\pi = v_1, \ldots, v_n$ be a linear order of $V(G)$. The *module-width* of π is the maximum, over all i, of the module number of $\{v_1, \ldots, v_i\}$. The *linear module-width* of G is the minimum module-width over all its linear orders.

Lemma 11. *Let G be a graph and $\mathcal{B}$ be a path decomposition of G of width w. Then one can construct in polynomial time and logarithmic space a linear order of module-width at most $w + 2$.*

Proof. For each vertex v, let B_v be the leftmost bag (i.e., the bag with the smallest index) of $\mathcal{B}$ containing v. Let $\pi = v_1, \ldots, v_n$ be a linear order of $V(G)$ such that the bags $B_{v_1}, \ldots, B_{v_n}$ appear in the same order as in $\mathcal{B}$, with ties broken arbitrarily. Clearly, this order can be constructed within the claimed time and space bounds; we argue that it has module-width at most $w + 2$. By the properties of a path decomposition, for each $i \in [n]$, there are at most $w + 1$ vertices in $\{v_1, \ldots, v_i\}$ that have a neighbor in $\{v_{i+1}, \ldots, v_n\}$. Therefore the module-number of each such $\{v_1, \ldots, v_i\}$ can be at most $w + 2$: $w + 1$ for the aforementioned vertices and one for the vertices without neighbors in $\{v_{i+1}, \ldots, v_n\}$. ☐

2.1.1 Linear mim-width

For a graph G and a linear order $\lambda = v_1, \ldots, v_n$ of $V(G)$, the *mim-width of* λ is the maximum, over all $i \in [n-1]$, of the size of an induced matching in the bipartite subgraph of G consisting of all edges that have one endpoint in $\{v_1, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$. The *linear mim-width* of a graph G is the minimum mim-width over all its linear orders. We observe that b-Coloring parameterized by linear mim-width is para-NP-complete.

Observation 12. b-Coloring is NP-complete on graphs of linear mim-width 2.

Proof. It is well-known that Graph Coloring is NP-complete on circular-arc graphs [18]. Moreover, Belmonte and Vatshelle [2] showed that given a circular-arc representation of a graph G, we can construct a linear order of G of mim-width at most 2 in polynomial time.

We can now give a trivial reduction from Graph Coloring to b-Coloring as follows. Let (G, k) be an instance of Graph Coloring, where G is a circular-arc graph, given with a circular-arc representation. We obtain G' from G by adding a k-clique. Then it is clear that G' has a b-coloring with k colors if and only if G has a proper coloring with at most k colors. Moreover, given the circular-arc representation of G, we construct a linear order λ of mim-width at most 2 using [2]. We append the vertices of the k-clique in any order at the end of λ to obtain a linear order λ' of G'. Clearly, the mim-width of λ' is again at most 2. □

2.1.2 Twin-width

We skip the definition of twin-width here, and refer the reader to [7].

Observation 13. For any $k \geq 3$, the problem of determining if a graph G has a b-coloring with k colors is NP-complete on graphs of twin-width at most 8.

Proof. We reduce from 3-Coloring on planar graphs, which is well-known to be NP-complete [17]. Let G be a planar graph, and $k \geq 3$. Let G_3 be the graph obtained from adding a triangle to G. Now, we let G' be a graph obtained from G_3 by adding a $(k-3)$-clique that is adjacent to all vertices in $V(G_3)$. Then, G has a 3-coloring if and only if G' has a b-coloring with k colors. Planar graphs have twin-width at most 8 [20], and it is easy to see that adding a triangle or universal vertices to a graph cannot increase the twin-width [7]. □

2.2 The class XNLP

We assume familiarity with the basic technical notions of parameterized complexity and refer to [13] for an overview. The class XNLP, introduced as $N[f \text{ poly}, f \text{ log}]$ by Elberfeld et al. [14], consists of the parameterized decision problems that given an n-bit input with parameter k can be solved by a non-deterministic algorithm that simultaneously uses at most $f(k)n^c$ time and at most $f(k) \log n$ space, where f is a computable function and c is a constant. We refer to [6, 14] for more details on this complexity class. Hardness in XNLP can be transferred via *parameterized logspace reductions* [14] which are parameterized reductions in the traditional sense [13] with the additional constraint of using only $f(k) + \mathcal{O}(\log n)$ space, where once again k is the parameter of the problem and n is the input size.

3 Pathwidth

We show b-Coloring is XNLP-complete via a reduction from the following problem which is known to be XNLP-complete when parameterized by the width of a given path decomposition of the input graph [4].

Circulating Orientation

Input:	Undirected graph G with edge weights $\mathbf{w}\colon E(G) \to \mathbb{N}$ given in unary.
Question:	Is there an orientation $\overrightarrow{G}$ of G such that for every vertex $v \in V(G)$: $\sum_{vx \in E(\overrightarrow{G})} \mathbf{w}(vx) = \sum_{xv \in E(\overrightarrow{G})} \mathbf{w}(xv)$?

Theorem 14. *b-*Coloring *parameterized by the width of a given path decomposition of the input graph is* XNLP*-complete.*

Proof. To show XNLP-hardness, we give a parameterized logspace-reduction from the Circulating Orientation problem parameterized by the width of a given path decomposition of the input graph, which was shown to be XNLP-complete in [4]. Let $(G, \mathbf{w})$ be an instance of Circulating Orientation, given with a path decomposition $\mathcal{B}$ of G. We let $n = |V(G)|$, $m = |E(G)|$, and $\mathbf{W} = \sum_{e \in E(G)} \mathbf{w}(e)$. For each vertex $v \in V(G)$, we let $W_v = \sum_{uv \in E(G)} \mathbf{w}(uv)$. We may assume that G is connected and that for all $e \in E(G)$, $\mathbf{w}(e) \geq 1$; therefore $\mathbf{W} \geq m \geq n - 1$.

We construct an equivalent instance (H, k) of b-Coloring. We let

$$k = 2\mathbf{W} + 2m + n + 2. \tag{1}$$

We begin the construction of H which is illustrated in Figure 2 by adding $2\mathbf{W} + 2$ disjoint copies of a star with $k - 1$ leaves. Let $S^\star$ be one of these stars. We denote its center by $s^\star$ and refer to it throughout the proof as the *superstar*. The remaining ones are referred to as *anonymous*. We partition a subset of the leaves of $S^\star$ into $\mathcal{L} = \{L_{e,v} \mid e \in E(G), v \in e\}$ where for all $e \in E(G)$ and $v \in e$, $|L_{e,v}| = \mathbf{w}(e)$. Note that this is possible since $k - 1 \geq 2\mathbf{W}$.

Vertex gadget. For each $v \in V(G)$, we add v, as well as a set P_v of $k - \frac{3}{2}W_v - 1$ independent vertices to H. We add all edges between v and P_v. Furthermore, for each edge $e \in E(G)$ such that $v \in e$, we connect v and the vertices in $L_{e,v}$ in H.

Edge gadget. For each $e = uv \in E(G)$, we add the following gadget to H. First, it has two vertices $x_{e,u}$ and $x_{e,v}$, a set Y_e of $\mathbf{w}(e)$ vertices, and a set Z_e of $k - 2\mathbf{w}(e) - 3$ vertices. The vertex $x_{e,u}$ is adjacent to $Y_e \cup Z_e \cup L_{e,u}$, and $x_{e,v}$ is adjacent to $Y_e \cup Z_e \cup L_{e,v}$. We make u and v adjacent to Y_e. We furthermore add one new vertex q_e to H as well as all edges between q_e and $Z_e \cup L_{e,u} \cup L_{e,v} \cup \{x_{e,u}, x_{e,v}\}$. We let $X = \{x_{e,u}, x_{e,v} \mid e = uv \in E(G)\}$, and $Q = \{q_e \mid e \in E(G)\}$.

Adding all vertex and edge gadgets finishes the construction of H, which can be performed using only logarithmic space.

Claim 14.1. If $(G, \mathbf{w})$ has a circulating orientation, then H has a b-coloring with k colors.

Proof. Let $\overrightarrow{G}$ be the circulating orientation of $(G, \mathbf{w})$. We give a coloring of the vertices of H with colors $[0..(k-1)]$. To do so, we identify some important subsets of $[0..(k-1)]$ whose b-vertices will appear in targeted regions of H. First, we let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. We construct a proper coloring of H such that once the coloring is completed, the following hold.

(i) The vertex $s^\star$ (the center of the superstar) is a b-vertex of color 0.

(ii) For each $i \in [n]$, v_i is a b-vertex of color i.

(iii) For each $i \in [m]$, q_{e_i} is a b-vertex of color $n + i$.

(iv) For each $i \in [m]$, either $x_{e_i,u}$ or $x_{e_i,v}$, where $e_i = uv$, is a b-vertex of color $m + n + i$.

(v) Each of the remaining $k - (2m + n + 1) = 2\mathbf{W} + 1$ colors has a b-vertex that is a center of an anomymous star.

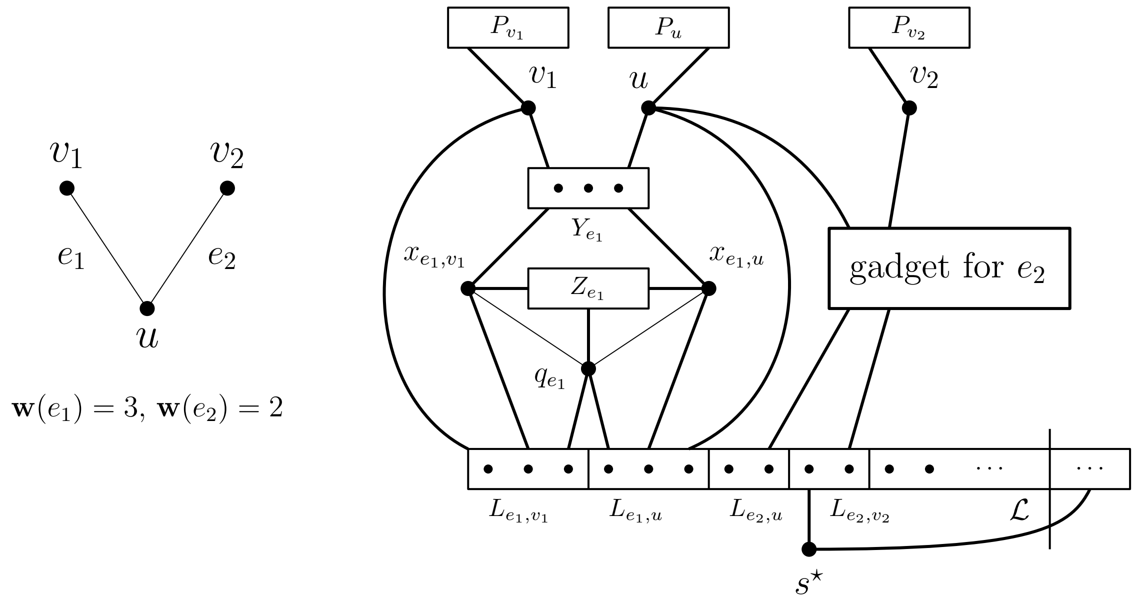

Figure 2: Sketch of the main part of the reduction. Bold edges mean that all edges between the corresponding sets are present. All vertex sets represented by single boxes are independent. Note that $|L_{e,v_1}| = |L_{e,u}| = |Y_{e_1}| = \mathbf{w}(e_1) = 3$ and recall that $|Z_{e_1}| = k - 2\mathbf{w}(e_1) - 3$.

Let $S_1, \ldots, S_{2\mathbf{W}+1}$ be the anonymous stars with centers $s_1, \ldots, s_{2\mathbf{W}+1}$, respectively. For each $i \in [2\mathbf{W}+1]$, we assign s_i the color $2m+n+i$, and the leaves of S_i the colors $[0..(k-1)] \setminus \{2m+n+i\}$. This satisfies (v). We assign $s^\star$ the color 0 and its leaves the colors $[k-1]$ in such a way that colors $[(2m+n+1)..(2m+n+2\mathbf{W})]$ appear on the vertices in $\mathcal{L}$. This satisfies (i). For each $i \in [n]$, we assign v_i color i. For each $i \in [m]$ and each $v \in e_i$, we let $C_{e_i,v}$ be the colors appearing on $L_{e_i,v}$, and we assign $x_{e_i,v}$ the color $m+n+i$.

We now color the edge gadgets. Let $i \in [m]$ and $e_i = uv$. We give q_{e_i} color $n+i$. We assign the vertices in Z_{e_i} the colors

$$[0..(k-1)] \setminus (C_{e_i,u} \cup C_{e_i,v} \cup \{n+i, m+n+i\}).$$

If e_i is directed from u to v in $\overrightarrow{G}$, then we repeat colors $C_{e_i,u}$ on Y_{e_i}. Observe that this makes $x_{e_i,v}$ a b-vertex for color $m+n+i$: it sees colors $C_{e_i,v}$ on $L_{e_i,v}$, colors $C_{e_i,u}$ on Y_{e_i}, and the remaining colors other than its own on $Z_{e_i} \cup \{q_{e_i}\}$. Moreover, q_{e_i} is a b-vertex for color $n+i$, since it sees color $m+n+i$ on $x_{e_i,v}$, and the remaining colors on $L_{e_i,u} \cup L_{e_i,v} \cup Z_{e_i}$. Once this is done for all i, (iii) and (iv) are satisfied.

We now color the vertex gadgets. We first argue that each $v \in V(G)$ already sees precisely $\frac{3}{2}W_v$ colors in its neighborhood. This is because v sees W_v colors on $\bigcup_{e \in E(G), v \in e} L_{e,v}$, and for each edge e that is directed towards v, there are $\mathbf{w}(e)$ additional colors appearing in the neighborhood of v; concretely, on the set Y_e of the corresponding edge gadget. Since $\overrightarrow{G}$ is circulating, the latter contribute with an additional $\frac{1}{2}W_v$ colors in total. Therefore, we can distribute the remaining $k - \frac{3}{2}W_v - 1$ colors on the set P_v, which makes v a b-vertex. This satisfies (ii), and we have arrived at a b-coloring of H with k colors. ⌟

We now work towards the reverse implication of the correctness proof. We start with a claim regarding the location of the b-vertices in any b-coloring of H with k colors. Throughout the following, we denote by A the set of centers of the anonymous stars.

Claim 14.2. Each b-coloring of H with k colors has precisely one b-vertex per color. Moreover, the b-vertices are $\{s^\star\} \cup V(G) \cup Q \cup A$, and for each $e = uv \in E(G)$, precisely one of $x_{e,u}$ and $x_{e,v}$.

Proof. The only vertices with high enough degree (at least $k-1$) to become b-vertices in such a coloring of H are in $\{s^\star\} \cup V(G) \cup Q \cup A \cup X$. Note that this set has size $2\mathbf{W} + 3m + n + 2 = k + m$.

We argue that the gadget of each edge $e = uv$ can contain at most two b-vertices. Note that only three of its vertices, $x_{e,u}$, $x_{e,v}$, and q_e have high enough degree to be b-vertices. Suppose for a contradiction that $x_{e,u}$ and $x_{e,v}$ are b-vertices for colors c_u and c_v, respectively, where $c_u \neq c_v$. For $x_{e,u}$ to be a b-vertex of color c_u, it needs to have a neighbor colored c_v. By the structure of H, this vertex has to be contained in $L_{e,u}$. Similarly, we can conclude that $L_{e,v}$ contains a vertex colored c_u. But this means that q_e has two neighbors colored c_u and two neighbors colored c_v. Since $\deg_H(q_e) = 2\mathbf{w}(e) + |Z_e| + 3 = k$, this means that each of these vertices sees at most $k-2$ colors in its neighborhood, so neither of them is a b-vertex, a contradiction.

Since we only have $k+m$ vertices of high enough degree to be b-vertices, we can only have enough b-vertices if each edge gadget has exactly three b-vertices, and if all vertices in $\{s^\star\} \cup V(G) \cup A$ are b-vertices. Now suppose that for some edge $e = uv \in E(G)$, both $x_{e,u}$ and $x_{e,v}$ are b-vertices for their color. By the structure of H, this implies that the same colors have to appear on $L_{e,u}$ and $L_{e,v}$. But $s^\star$ needs to be a b-vertex, now there are $\mathbf{w}(e) \geq 1$ colors in its neighborhood that repeat. Since $\deg_H(s^\star) = k-1$, this is not possible. This yields the claim. ⌟

Throughout the following, we assume that we have a b-coloring of H with k colors. Again, for each $e \in E(G)$ and $v \in e$, we denote by $C_{e,v}$ the set of colors appearing on the vertices $L_{e,v}$. We prove another auxiliary claim.

Claim 14.3.

(i) For each $e, e' \in E(G)$ and $v \in e$, $v' \in e'$, if $(e,v) \neq (e',v')$, then $C_{e,v} \cap C_{e',v'} = \emptyset$.

(ii) For each $e = uv \in E(G)$, either colors $C_{e,u}$ or colors $C_{e,v}$ appear on Y_e; the former if $x_{e,v}$ is a b-vertex and the latter if $x_{e,u}$ is a b-vertex.

Proof. (i). By Claim 14.2, we know $s^\star$ is a b-vertex. Since its degree is $k-1$, all its neighbors must receive distinct colors. Hence (i) follows.

(ii). By Claim 14.2, either $x_{e,v}$ or $x_{e,u}$ is a b-vertex for its color. Suppose that $x_{e,v}$ is a b-vertex for color i (the other case is analogous). For $x_{e,v}$ to be a b-vertex, the colors $C_{e,u}$ have to appear in its neighborhood. We show that the colors $C_{e,u}$ have to appear on Y_e, which yields the claim. By Claim 14.3(i), we have that $C_{e,u} \cap C_{e,v} = \emptyset$, so the colors $C_{e,u}$ have to appear on $Z_e \cup Y_e$. We rule out that they appear on Z_e. For the following argument, recall that by Claim 14.2, q_e is a b-vertex for its color; moreover, its degree is k, so it sees exactly one color twice in its neighborhood.

We distinguish two cases based on the color of the vertex $x_{e,u}$. Supposed $x_{e,u}$ received color i as well. Then, q_e sees color i twice, meaning that all remaining colors appear exactly once on its neighborhood. Since $Z_e \cup L_{e,u} \subset N(q_e) \setminus \{x_{e,u}, x_{e,v}\}$, no color from $C_{e,u}$ appears on Z_e. Now suppose that $x_{e,u}$ received a color $j \neq i$. Since $x_{e,v}$ is a b-vertex for color i, and since $\deg(x_{e,v}) = k-1$, there is precisely one vertex with color j in $N(x_{e,v})$. Since the given coloring of H is proper, this vertex cannot be in $Y_e \cup Z_e \cup \{q_e\} \subseteq N(x_{e,u})$. Therefore, there is a vertex of color j in $L_{e,v}$. This means that q_e sees color j twice, once on $x_{e,u}$ and once on a vertex in $L_{e,v}$. Subsequently, the vertices in $L_{e,u} \cup Z_e$ all receive unique colors, implying once again that no color from $C_{e,u}$ appears on Z_e. In either case, the only way that $x_{e,v}$ sees colors $C_{e,u}$ is if they appear on Y_e, which proves the claim. ⌟

We now construct an orientation $\overrightarrow{G}$ of G. For each edge $e = uv \in E(G)$, if $x_{e,u}$ is a b-vertex, then we orient e towards u, and if $x_{e,v}$ is a b-vertex, we orient e towards v. Note that by Claim 14.2, this is well-defined. Throughout the following whenever we write "uv" for an edge in $\overrightarrow{G}$, we mean that the edge uv is directed from u to v in $\overrightarrow{G}$. The next claim completes the correctness proof of the reduction.

Claim 14.4. For each $v \in V(G)$, $\sum_{uv \in E(\overrightarrow{G})} \mathbf{w}(uv) = \frac{1}{2}W_v$.

Proof. We first show that $\sum_{uv \in E(\overrightarrow{G})} \mathbf{w}(uv) \geq \frac{1}{2}W_v$. By Claim 14.2, v is a b-vertex. Moreover, $\deg_H(v) = k + \frac{1}{2}W_v - 1$ since v has $k - \frac{3}{2}W_v - 1$ neighbors in P_v, W_v additional neighbors in the edge gadgets, W_v additional neighbours in $\mathcal{L}$, and no other neighbors. This means that for v to be a b-vertex, v needs to see at least $\frac{1}{2}W_v$ colors in $\bigcup_{e \in E(G), v \in e} Y_e$. Claim 14.3 then implies that there is a set of edges $\{e_1, \ldots, e_d\}$ incident with v and with $\sum_{i \in [d]} \mathbf{w}(e_i) \geq \frac{1}{2}W_v$ such that for all $i \in [d]$, $x_{e_i,v}$ is a b-vertex. This implies the inequality by our construction of $\overrightarrow{G}$.

Now we show that $\sum_{uv \in E(\overrightarrow{G})} \mathbf{w}(uv) \leq \frac{1}{2}W_v$. Let $\mathcal{Y} = \bigcup_{e \in E(G)} Y_e$, note that $|\mathcal{Y}| = \mathbf{W}$, and that to make each $v \in V(G)$ a b-vertex, $\frac{1}{2}W_v$ colors must appear in $N_H(v) \cap \mathcal{Y}$ that are not in $N_H(v) \setminus \mathcal{Y}$. Moreover, for each $e = uv \in E(G)$, Y_e has colors that appear in $N_H(u) \setminus \mathcal{Y}$ but not in $N_H(v) \setminus \mathcal{Y}$ or vice versa by Claim 14.3. Since $\mathbf{W} = \sum_{e \in E(G)} \mathbf{w}(e) = \sum_{v \in V(G)} \frac{1}{2}W_v$, we can conclude that if for some $v \in V(G)$, $\sum_{uv \in E(\overrightarrow{G})} \mathbf{w}(uv) > \frac{1}{2}W_v$, then there is another $v' \in V(G) \setminus \{v\}$ with $\sum_{uv' \in E(\overrightarrow{G})} \mathbf{w}(uv') < \frac{1}{2}W_{v'}$, contradicting the previous paragraph. ⌟

Claim 14.5. Given a path decomposition of G of width w, one can construct a path decomposition of H of width at most $w + 5$ in polynomial time and logarithmic space.

Proof. Let $\mathcal{B}$ be a path decomposition of G of width w. We add $s^\star$ to all bags of $\mathcal{B}$. For each vertex $v \in V(G)$, let $B_v \in \mathcal{B}$ be a bag containing v. We insert a sequence of $|P_v|$ bags after B_v containing B_v, and a unique vertex of P_v. For each edge $e = uv \in E(G)$, let B_e be a bag in $\mathcal{B}$ containing u and v. We insert a sequence of $|Y_e \cup Z_e \cup L_{e,u} \cup L_{e,v}|$ bags after B_e containing B_e, $x_{e,u}$, $x_{e,v}$, q_e, and a unique vertex of $L_{e,u} \cup L_{e,v} \cup Y_e \cup Z_e$. Finally, we append a sequence of bags forming a width-1 path decomposition of the anonymous stars. Note that this gives a path decomposition of H and there is no bag to which we added more than six vertices. It is easy to see that these operations can be performed within the claimed time and space requirements. ⌟

Adapting the XP-algorithm for b-Coloring parameterized by module-width w [22] to a nondeterministic FPT-time and $f(w) \log n$ space algorithm, we can show that b-Coloring parameterized by linear module-width, and therefore by pathwidth, belongs to XNLP. This can be done similarly as in the case of Graph Coloring parameterized by linear clique-width as shown in [5]. This is summarized in the statement below.

Claim 14.6. b-Coloring parameterized by the module-width of a given linear order of the vertices of the input graph is in XNLP.

Membership then follows from Lemma 11 and Claim 14.6. This concludes the proof of the theorem. □

4 Neighborhood Diversity

In this section, we consider the parameterization by neighborhood diversity. We follow the same strategy as the one that Koutecký [24] applied for the Graph Coloring problem, that is, we give an ILP-formulation that can be solved efficiently by a parameterized ILP-algorithm due to Jansen and Rohwedder [23].[1]

Theorem 15 (Jansen and Rohwedder [23]). *For $A \in \mathbb{Z}^{r\times n}$, $b \in \mathbb{Z}^r$, $c \in \mathbb{Z}^n$, the ILP*[2]

$$\min\{c^T x \colon Ax = b, x \in \mathbb{Z}^n_{\geq 0}\}$$

can be solved in time $\mathcal{O}((\sqrt{r}\Delta)^{2r}) \cdot \log \|b\|_\infty + \mathcal{O}(rn)$, where $\Delta = \|A\|_\infty = \max_{i,j} A(i,j)$ and $\|b\|_\infty = \max_i b(i)$.

Note that in the previous theorem, the number of rows r in the ILP is equal to the number of constraints. Recall that an optimal ND-partition can be computed in polynomial time (Remark 6).

Theorem 16. b-Coloring *parameterized by the neighborhood diversity d of the input n-vertex graph is fixed-parameter tractable. Given an ND-partition of the input graph, the algorithm runs in time $2^{\mathcal{O}(d\log d)} \log n + \mathcal{O}(n)$.*

Proof. Suppose we want to find a b-coloring with k colors. Let G be a graph of neighborhood diversity at most d with ND-partition $\mathcal{P} = (P_1, \ldots, P_d)$. We create another partition $\mathcal{P}'$ of $V(G)$ as follows. For each P_i that is an independent set of size at least two, we pick one vertex v_i, let $P_i' = P_i \setminus \{v_i\}$, remove P_i from $\mathcal{P}$ and add P_i' and $\{v_i\}$ to $\mathcal{P}'$. All other parts of $\mathcal{P}$ are added to $\mathcal{P}'$ as they are. As a convention, we consider each $\{v_i\}$ a clique of size 1, and each such part in $\mathcal{P}'$ a clique part. Note that $d' = |\mathcal{P}'| \leq 2d$.

We start with a few observations.

(i) If $u, v \in V(G)$ are false twins, then in each proper coloring of G, either both u and v are b-vertices for the same color, or neither of them is a b-vertex.

(ii) For each $P \in \mathcal{P}'$ that is a clique, either all vertices in P are b-vertices for their color, or none of them are.

(iii) In each b-coloring of G, each color class has a b-vertex contained in a clique part of $\mathcal{P}'$.

(i) is immediate, (ii) follows from the fact that all vertices that are in the same part are twins, and (iii) follows from (i) and our construction: if there was an independent part with more than one vertex in $\mathcal{P}$, we split off a single vertex into a new part, which is now considered a clique part. If in a b-coloring, some independent part (of the original ND-partition $\mathcal{P}$) had a b-vertex, then the split off vertex is a b-vertex for the same color by (i), considered a clique part in $\mathcal{P}'$.

Next, we guess which clique parts of $\mathcal{P}' = (P_1, \ldots P'_{d'})$ contain b-vertices in the solution we are looking for. From now on, fix one such choice $B \subseteq [d']$.

We construct an ILP as follows. Let $H = G/\mathcal{P}'$. Each color class is described by its *type*, that is, the parts of $\mathcal{P}'$ it intersects. Note that each type is an independent set in H. Therefore, for each independent set I in H, we add a variable x_I, which counts how many color classes of that type there are. From now on, we denote by $\mathcal{I}(H)$ the independent sets of H. Now, the sum, over

[1]Note that the arXiv-version contains an improved running time over the version published in the ITCS 2019 proceedings.

[2]For a vector c, c^T denotes its transpose.

all independent sets I of H of the x_I will correspond to the total number of colors used. We add a constraint that ensures that this number is k. Moreover, for each clique part P'_i, we have to make sure that exactly $|P'_i|$ colors appear on that part, and in each independent part $P'_{i'}$, at least one color must appear. Finally, we have to ensure that there are k b-vertices. Note that, since all b-vertices are clique parts, by Observation (ii), each vertex in each part P'_i for $i \in B$, has to be a b-vertex. Therefore, for each $i \in B$, we ensure that the number of colors intersecting the closed neighborhood of vertex i in H is equal to k. To ensure that each color class has a b-vertex, we use the objective function to minimize the number of color classes that do not intersect B. If this value is 0, then we have a b-coloring, otherwise not. The ILP is:

$$\begin{aligned}
\min & \sum\nolimits_{I \in \mathcal{I}(H), I \cap B = \emptyset} x_I \\
\text{s.t.} & \sum\nolimits_{I \in \mathcal{I}(H)} x_I = k \\
& \sum\nolimits_{I \in \mathcal{I}(H), i \in I} x_I = |P'_i|, \qquad \text{if } P'_i \text{ is a clique} \\
& \sum\nolimits_{I \in \mathcal{I}(H), i \in I} x_I \geq 1, \qquad \text{if } P'_i \text{ is an independent set} \\
& \sum\nolimits_{I \in \mathcal{I}(H), I \cap N_H[i] \neq \emptyset} x_I = k, \text{ if } i \in B
\end{aligned} \tag{2}$$

The correctness of this formulation follows fairly straightforwardly from the discussion above.

Observation 16.1. The previous ILP has a solution with value 0 *if and only if G has a b-coloring with k colors whose b-vertices intersect precisely the parts $\{P'_i \mid i \in B\}$.*

For each guess of B, we construct an ILP as above. If there is one guess for which we have a solution with value 0, we report that G has a b-coloring with k colors, and say No otherwise. Correctness directly follows from Observation 16.1.

Let us analyze the run time. We can obtain the ND-partition $\mathcal{P}'$ from $\mathcal{P}$ in time $\mathcal{O}(n)$. We can then compress $\mathcal{P}'$ to remember only $|P'_i|$ for each $i \in [d']$ and one representative vertex per P'_i, also in time $\mathcal{O}(n)$. We solve $2^{\mathcal{O}(d)}$ many ILPs using Theorem 15 with $\mathcal{O}(d)$ rows and $2^{\mathcal{O}(d)}$ variables. From the compressed representation, each such ILP can be constructed in $2^{\mathcal{O}(d)} \log n$ time. Note that the inequalities (2) can be turned into equalities by adding at most $2^{\mathcal{O}(d)}$ slack variables. In the resulting ILP, the largest coefficient of any variable is 1, and the largest value on the right-hand side is at most n, since we may assume that $k \leq n$, and clearly, for each $i \in [d']$, $|P'_i| \leq n$. Therefore, we have $\Delta = 1$ and $\|b\|_\infty \leq n$, and each of the ILPs can be solved in time

$$2^{\mathcal{O}(d \log d)} \log n + d \cdot 2^{\mathcal{O}(d)} = 2^{\mathcal{O}(d \log d)} \log n,$$

yielding the claimed run time bound. □

5 Twin Cover

In this section, we prove the following theorem. This will be done by reducing the input graph of bounded twin cover number to a graph of bounded neighborhood diversity and then applying the algorithm from Theorem 16.

Theorem 17. b-Coloring *parameterized by the twin cover number of the input graph is fixed-parameter tractable. Given a graph with n vertices, m edges, and twin cover number t, it is solvable in $2^{2^{\mathcal{O}(t)}} n + \mathcal{O}(m)$ time.*

Let (G,k) be an instance of b-Coloring. If the size of a minimum twin-cover of G is at most t, then one can compute a twin-cover S of G of size at most t in $\mathcal{O}(1.2378^t + tn + m)$ time [16]. Recall that each connected component of $G - S$ consists of a clique C consisting of twins (Observation 8). Since C is a clique, we may assume $|C| \le k$.

Without loss of generality, let the color set be $\{1,\dots,k\}$ and assume the vertices of S get colors from the set $\{1,\dots,\min\{k,t\}\}$. For each $\texttt{col} : S \to \{1,\dots,\min\{k,t\}\}$ such that $\texttt{col}$ is a proper coloring of $G[S]$, perform the following steps. Note, in the following, $\texttt{col}$ is fixed.

For each $A \subset S$, let $\mathcal{C}_A = \{C : C \text{ is a maximal clique of } G-S \text{ and } N(C) = A\}$. Throughout, we use the shorthand $\bigcup \mathcal{C}_A$ for $\bigcup_{C \in \mathcal{C}_A} C$. Let $c_A^{\texttt{col}}$ denote the number of distinct colors used by the vertices of A in $\texttt{col}$, that is $c_A^{\texttt{col}} = |\cup_{a\in A} \texttt{col}(a)|$. Note that if a b-coloring with k colors of G coincides with $\texttt{col}$ on the vertex set S, then for any $C \in \mathcal{C}_A$, $k \ge |C| + c_A^{\texttt{col}}$. Thus, if this inequality does not hold, then $\texttt{col}$ cannot be extended into a k-b-coloring of G. In this case, discard $\texttt{col}$ and consider the next available (non-discarded) proper coloring function on S. Henceforth, assume that for each $C \in \mathcal{C}_A$, $|C| \le k - c_A^{\texttt{col}}$.

Observation 18. Let C be a maximal clique of $G-S$. A vertex $v \in C$ is a b-vertex of some b-coloring of G with k colors that extends $\texttt{col}$, if and only if $|C| = k - c_A^{\texttt{col}}$.

Proof. If $|C| = k - c_A^{\texttt{col}}$, then in any proper coloring of G with k colors that extends $\texttt{col}$, and for any color $i \in \{1,\dots,k\}$, there exists a vertex of $N[C]$ that gets color i. Since $N(v) = N[C] \setminus \{v\}$, v is a b-vertex.

In the other direction, assume $|C| \le k - c_A^{\texttt{col}} - 1$. Then for any $v \in C$, since $N(v) = N[C] \setminus \{v\}$, in any proper coloring of G that extends $\texttt{col}$, the neighbours of v get colors from a color set of size strictly less than $|C \setminus \{v\}| + c_A^{\texttt{col}}$, which is strictly less than $k-1$. Therefore, v cannot be a b-vertex. ⌟

For each $A \subseteq S$, let $C_A^{\max} \in \mathcal{C}_A$ denote a clique of maximum cardinality among the cliques in $\mathcal{C}_A$. Note that for any $v \in \bigcup \mathcal{C}_A \setminus C_A^{\max}$, by Observation 18, if (G,k) is a Yes-instance of b-Coloring witnessed by a coloring that extends $\texttt{col}$, then there exists a k-b-coloring that extends $\texttt{col}$ where, if v is a b-vertex, then it is not the unique b-vertex of its color, as $C_A^{\max}$ would also contain one such vertex. This is the idea behind the next reduction rule, which deletes vertices until the number of vertices in all the cliques of $\mathcal{C}_A$ is bounded.

Reduction Rule 19. If there exists $A \subseteq S$ such that $|\bigcup \mathcal{C}_A| \ge k - c_A^{\texttt{col}} + 1$, then let $v \in \bigcup \mathcal{C}_A \setminus C_A^{\max}$ and delete v from the graph.

Lemma 20. *Reduction Rule 19 is safe, i.e., under its preconditions, G has a b-coloring with k colors if and only if $G \setminus v$ has a b-coloring with k colors.*

Proof. In both directions, let $C \in \mathcal{C}_A$ denote the clique containing v. Consider the backward direction first. That is, assume $G \setminus v$ has a b-coloring $\texttt{col}^\dagger$ with k colors that extends $\texttt{col}$. Recall that $|C| \le k - c_A^{\texttt{col}}$. Since $N(v) = N[C] \setminus \{v\}$, the number of distinct colors that appear on $N(v)$ in $\texttt{col}^\dagger$ is at most $|C \setminus \{v\}| + c_A^{\texttt{col}} \le k - c_A^{\texttt{col}} - 1 + c_A^{\texttt{col}} \le k-1$. Then, there exists a color $i \in \{1,\dots,k\}$ that does not appear on $N(v)$ in $\texttt{col}^\dagger$. Thus extending $\texttt{col}^\dagger$ by coloring v with the color i yields a b-coloring of G.

Now consider the forward direction. Let $\texttt{col}^\dagger$ be a b-coloring of G with k colors that extends $\texttt{col}$, such that if v is a b-vertex, then v is not the unique b-vertex of its color. We argue that such a coloring exists. Indeed, by Observation 18, if v belongs to a clique of size strictly smaller than $k - c_A^{\texttt{col}}$, then v is not a b-vertex. In the other case, $v \notin C_A^{\max}$, which means that C and $C_A^{\max}$ must have the same size. Since v is a b-vertex, say of color i, all colors appear on $N[v] = C \cup A$, and therefore all colors appear on $C_A^{\max} \cup A$, with the colors on C repeating on $C_A^{\max}$. This implies that there is a b-vertex of color i in $C_A^{\max}$ as well.

Now we show that after deleting v we still have a b-coloring of $G \setminus v$. Suppose $\mathtt{col}^\dagger(v) = i$. If v is a b-vertex of its color, then by assumption there is another b-vertex of color i. Now suppose that v was used to make another vertex u of color j a b-vertex. Suppose further that v is the unique neighbour of u among all vertices that are colored i by $\mathtt{col}^\dagger$. Suppose $u \in C$. This implies that $|C| = |C_A^{\max}|$ and by the same argument as above, color j has a b-vertex in $C_A^{\max}$ as well. If $u \notin C$, then $u \in S$. Since $|\bigcup \mathcal{C}_A| \geq k - c_A^{\mathtt{col}} + 1$ and $\mathtt{col}^\dagger$ is a coloring with k colors, by the pigeonhole principle, there exist $v', v'' \in \bigcup \mathcal{C}_A$ that have the same color. Let $\mathtt{col}^\dagger(v') = \mathtt{col}^\dagger(v'') = j'$. Since v' and v'' have the same color, we may assume $v' \notin C_A^{\max}$. Construct $\mathtt{col}'^\dagger : V(G) \setminus \{v\} \to \{1, \ldots, k\}$ such that $\mathtt{col}'^\dagger(x) = \mathtt{col}^\dagger(x)$ for all $x \neq v'$, and $\mathtt{col}'^\dagger(v') = i$. We now observe that $\mathtt{col}'^\dagger$ is a proper coloring of $G \setminus \{v\}$, since v was the unique vertex in $\bigcup \mathcal{C}_A$ colored i. Moreover, every color still has a b-vertex. ⌟

Consider the instance obtained after the exhaustive application of Reduction Rule 19. For brevity of notation let the instance be $(G, k, S, \mathtt{col})$ where S is a twin cover of G of size at most t and $\mathtt{col}$ is a proper coloring of $G[S]$ with colors from $\{1, \ldots, \min\{k, t\}\}$. Further, for each $A \subseteq S$, the number of vertices present in the union of the cliques in $\mathcal{C}_A$ is at most $k - c_A^{\mathtt{col}}$ and therefore $\mathcal{C}_A$ contains at most one clique of size $k - c_A^{\mathtt{col}}$. We call such an instance a *cleaned* instance.

Lemma 21. *If a cleaned instance $(G, k, S, \mathtt{col})$ is a* Yes*-instance, then there exists a k-b-coloring where for each $A \subseteq S$, the vertices of $\bigcup \mathcal{C}_A$ get distinct colors.*

Proof. Let $\mathtt{col}^\dagger$ be a solution of the instance $(G, k, S, \mathtt{col})$. For each $A \subseteq S$ proceed as follows. Recall that $|\bigcup \mathcal{C}_A| \leq k - c_A^{\mathtt{col}}$. Also the neighbors of $\bigcup \mathcal{C}_A$ in G (which is the set A) intersect $c_A^{\mathtt{col}}$ many color classes of $\mathtt{col}^\dagger$. Let this color set be $I_A \subseteq \{1, \ldots, k\}$. Consider the clique $C_A^{\max} \in \mathcal{C}_A$. Let the color set of $C_A^{\max}$ in $\mathtt{col}^\dagger$ be I_1. Then $I_A \cap I_1 = \emptyset$. Consider the vertices in $\bigcup \mathcal{C}_A \setminus C_A^{\max}$. Note that, in this case, the vertices of $\bigcup \mathcal{C}_A \setminus C_A^{\max}$ are not b-vertices. Indeed, in a cleaned instance, either a b-vertex belongs to $C_A^{\max}$, in which case $\bigcup \mathcal{C}_A \setminus C_A^{\max} = \emptyset$, or a b-vertex belongs to S. Thus, if there exists two vertices $v, v' \in \bigcup \mathcal{C}_A \setminus C_A^{\max}$, such that $\mathtt{col}^\dagger(v) = \mathtt{col}^\dagger(v')$, then arbitrarily assign a color to v' that is not assigned to any of its neighbours. Repeat this, until all vertices of $\bigcup \mathcal{C}_A \setminus C_A^{\max}$ get distinct colors (and their colors are disjoint from $I_A \cup I_1$). It is easy to see that the resulting coloring is also a b-coloring of G that respects $\mathtt{col}$. ⌟

The safeness of the following reduction rule follows from Lemma 21.

Reduction Rule 22. For each $A \subseteq S$, delete the cliques in $\mathcal{C}_A \setminus \{C_A^{\max}\}$ and add a new clique of size $|\bigcup \mathcal{C}_A \setminus C_A^{\max}|$ whose neighbourhood in G is exactly A.

Lemma 23. *When Reduction Rules 19 and 22 are no longer applicable, the neighbourhood diversity of G is at most $2^{t+1} + t$.*

Proof. Since Reduction Rule 19 is no longer applicable, we can assume that the given instance is a clean instance. From Reduction Rule 22, for each $A \subseteq S$, the number of cliques of $G - S$ is at most two. Therefore the total number of cliques of $G - S$ is at most 2^{t+1}. Thus the neighbourhood diversity of G is at most $2^{t+1} + t$, where the ND-partition is composed of the cliques in $G - S$ plus $t = |S|$ many sets of size one, each containing a vertex of S. ⌟

Proof (of Theorem 17). The algorithm starts by finding a twin-cover of G of size t in $\mathcal{O}(1.2378^t + tn + m)$ time [16]. The algorithm guesses the restriction of the k-b-coloring of G onto the vertices of S. Assuming that the colors of S are from the set $\{1, \ldots, \min\{k,t\}\}$ in the k-b-coloring, the number of guesses is at most $t! = 2^{\mathcal{O}(t \log t)}$. Applying Reduction Rule 19 on this instance, we get a cleaned instance. Then applying Reduction Rule 22 on this instance, by Lemma 23 we conclude that the neighbourhood diversity of G is at most $2^{t+1} + t$. Applying all reduction rules can be done in time $\mathcal{O}(2^t \cdot n)$. Using the algorithm of Theorem 16, we solve the problem in $2^{2^{\mathcal{O}(t)}} \cdot \log n + \mathcal{O}(n)$ time. If for neither of the guessed colorings of S, the above algorithm reports Yes, then report a No. Otherwise, report Yes. $\square$

6 Conclusion

We explored the landscape of structural parameterizations of b-Coloring. We showed that the problem is XNLP-complete parameterized by pathwidth, which implies it is W[t]-hard for any t by pathwidth, and as a consequence, by treewidth and clique-width as well. Recall that b-Coloring was already known to be XP parameterized by clique-width. The algorithm of [22] runs in time $n^{2^{O(w)}}$, where w is the clique-width of the input graph (which is tight under the Exponential Time Hypothesis). Since graphs of treewidth t have clique-width $2^{\Theta(t)}$ [12], this results in an XP algorithm for b-Coloring parameterized by treewidth with running time $n^{2^{2^{O(t)}}}$. It would be interesting to investigate if this dependence on the treewidth can be improved, and accompanied by a matching lower bound under the ETH.

On the positive side, we showed b-Coloring to be FPT parameterized by neighborhood diversity and twin cover, two generalizations of vertex cover to more dense graphs. A parameter that generalizes both neighborhood diversity and twin cover is modular-width, defined by Gajarský, Lampis and Ordyniak [15]. The complexity of b-Coloring parameterized by modular-width remains an interesting open problem.

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