A Logic-Based Algorithmic Meta-Theorem for Mim-Width

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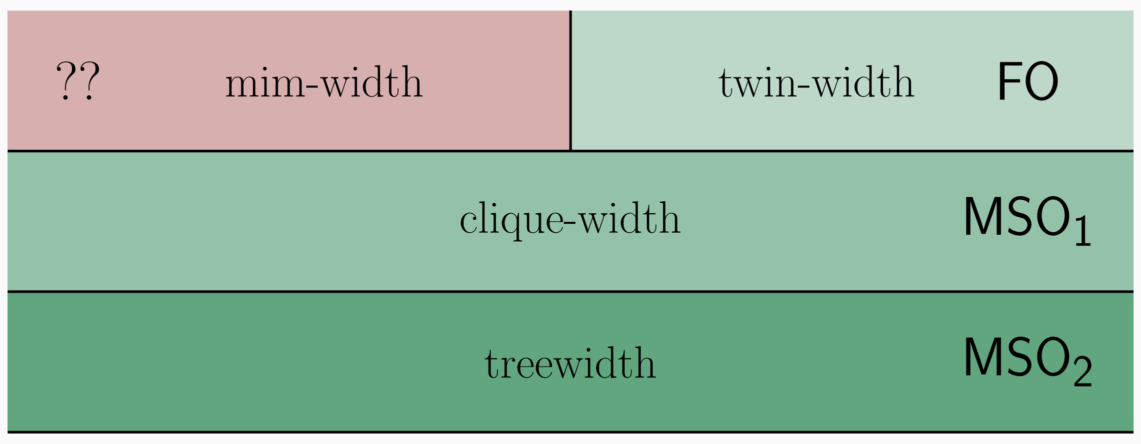
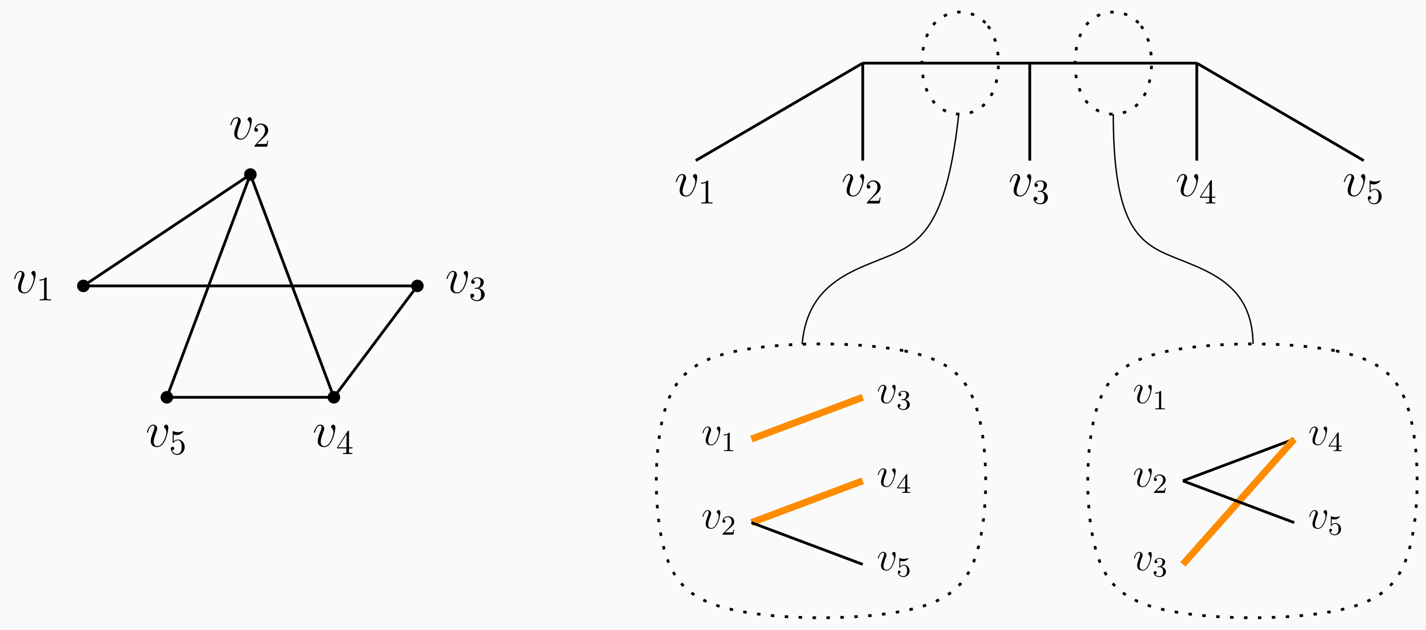
∀ Highlights 2022

Algorithmic Meta-Theorems

"Each problem expressible in logic L is efficiently solvable on graphs of bounded *-width."

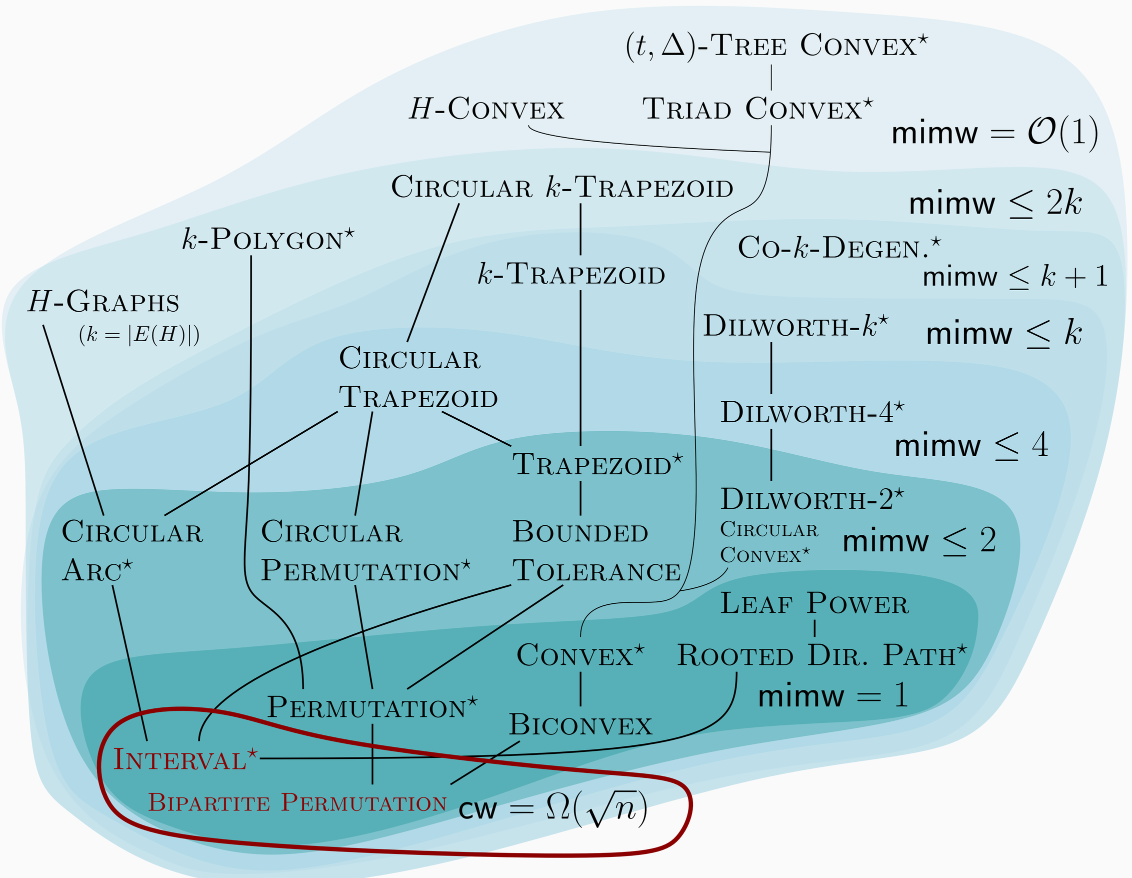


Branch decompositions and mim-width

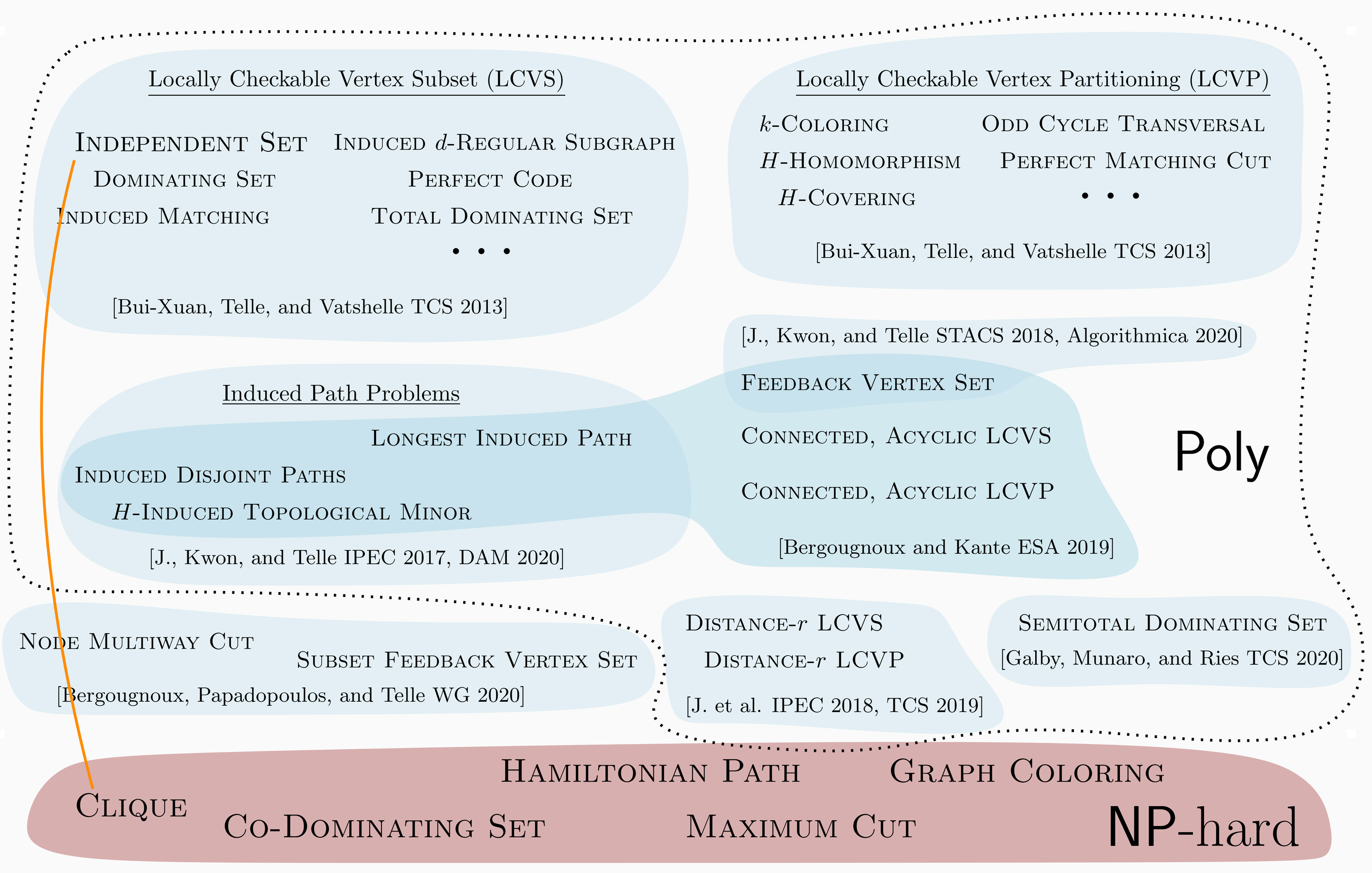


- Mim-width of branch decomposition: Max mim-value over all cuts.
- Mim-width of graph: Min mim-width over all its branch decompositions.

Graph classes of bounded mim-width



Mim-width and algorithmic problems



Existential MSO_1

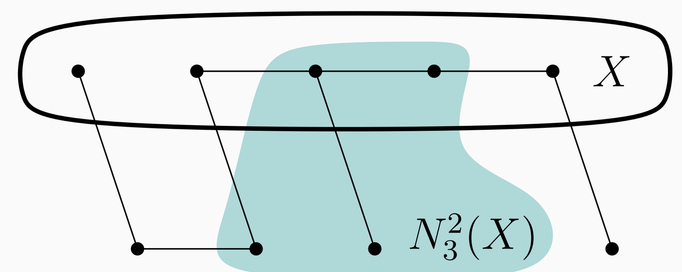
- Vertex variables $x, y, z, \dots$ and vertex set variables $X, Y, Z, \dots$
- Vertex set constants (colors) $\mathbf{P}, \mathbf{Q}, \dots$
- Connectives: and ($\wedge$), or ($\vee$), negation ($\neg$) only on quantifier-free formulas.
- $\exists x$, $\exists X$, $x = y$, $x \in X$, $E(x, y)$ (adjacency)

Example: $\exists x_1 \dots \exists x_k \bigwedge_{i \neq j} \neg E(x_i, x_j)$

Neighborhood operator

$N^r_d(X)$ is the set of vertices v at distance at most r to at least d vertices in $X \setminus \{v\}$.

$N^2_3(X)$



Neighborhood terms

...are built from set variables $(X, Y, Z, \dots)$ or set constants $(\mathbf{P}, \mathbf{Q}, \dots)$, by applying

- the neighborhood operator $N^r_d(t)$ for a neighborhood term t, or
- basic set operations $t_1 \cup t_2$, $t_1 \cap t_2$, $t_1 \setminus t_2$, $\overline{t_1}$ for neighborhood terms t_1, t_2.

A&C distance neighborhood (DN) logic

...is the extension of Existential MSO_1 with

- size measurements of nbh. terms t: $|t| \le m$ or $|t| \ge m$,
- comparison between nbh. terms t_1, t_2: $t_1 \subseteq t_2$ or $t_1 = t_2$,
- connectivity constraints on nbh. terms t: $\mathsf{conn}(t)$, and
- acyclicity constraints on nbh. terms t: $\mathsf{acy}(t)$.

Examples

- Max. Independent Set: $\exists X\ |X| \ge m \wedge N^1_1(X) \cap X = \emptyset$
- Min. Dominating Set: $\exists X\ |X| \le m \wedge \overline{X \cup N^1_1(X)} = \emptyset$
- Min. Feedback Vertex Set: $\exists X\ |X| \le m \wedge \mathsf{acy}(\overline{X})$
- Longest Induced Path: $\exists X\ |X| \ge m \wedge \mathsf{conn}(X) \wedge \mathsf{acy}(X) \wedge X \setminus N^1_3(X) = X$

Theorem

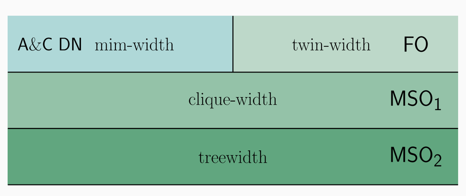
There is a model checking algorithm for A&C DN logic that runs in

$$n^{\mathcal{O}(dw|\varphi|^2)} \text{ time,}$$

- where $d = d(\varphi)$ is the maximum such that $N^{\cdot}_d(\cdot)$ appears in φ, and
- w is the mim-width of a given branch decomposition.

Our meta-theorem...

- ...captures almost all problems solvable in polynomial time on graphs of bounded mim-width (see in the dotted box to the left), and more!
- ...also implies single-exponential algorithms param. by tree-width and clique-width.
- ...is (nearly) tight in terms of run time and in several senses, expressive power.



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