

# Status of Open Problems in the Thesis “Bounded Width Graph Classes in Parameterized Algorithms”

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## Abstract

In this document I keep track of the status of the open problems I mentioned in my thesis. If you are aware of any results I have missed or solved one of the problems, I would appreciate it if you got in touch with me.

## 1 The List

#	Description	Status	Comments
3.1	Is there an algorithm for TREE DECOMPOSITION running in time $2^{o(k^3)} \cdot n^{O(1)}$ ?	solved	Yes. (E)
3.2	Is there some function $f: \mathbb{N} \rightarrow \mathbb{N}$ and an algorithm that given a graph $G$ and an integer $k$ , either decides that $\text{mimw}(G) > k$ , or outputs a branch decomposition of $G$ of mim-width at most $f(k)$ , and runs in XP time parameterized by $k$ ?	open	(H)
3.3	Is there some constant $c$ and an algorithm that given a graph $G$ , either decides that $\text{mimw}(G) > 1$ , or outputs a branch decomposition of $G$ of mim-width at most $c$ , and runs in polynomial time?	open	
3.4	Is there some fragment of MSO/First Order Logic of graphs, containing a problem that is not locally checkable, and whose corresponding MODEL CHECKING problem is solvable in time $n^{f( \phi , \text{mimw})}$ , if a branch decomposition of mim-width $\text{mimw}$ of the input graph is provided?	solved	A&C DN (B)
4.1	For constant $q \geq 8$ , is the clique-width of $(q, q - 2)$ -graphs bounded by a constant or not?	open	

4.2	Is there a polynomial-time algorithm for MAXIMUM INDEPENDENT SET when the input graph is given together with one of its branch decompositions of sim-width 1?	open	
9.1	Characterize the graph class LINEAR MIM-WIDTH 1.	open	
9.2	Are all connected, acyclic (co-) $(\sigma, \rho)$ -problems either polynomial-time solvable or W[1]-hard when parameterized by the mim-width of a given (linear) branch decomposition of the input graph?	open	(C)
9.3	Would an $n^{o(w)}$ -time algorithm for some $(\sigma, \rho)$ -problem parameterized by the mim-width $w$ of a given (linear) branch decomposition of the input graph refute ETH?	solved	Yes. (F)
10.1	Is FALL COLORING parameterized by clique-width W[1]-hard?	solved	Yes. (A)
10.2	Would an $n^{2^{o(w)}}$ -time algorithm for FALL COLORING, where $w$ denotes the clique-width of the input graph, refute ETH?	solved	Yes. (A)
10.3	Is CLIQUE COLORING parameterized by clique-width W[1]-hard?	solved	No, it is FPT (see 10.4)
10.4	Is there a computable function $g: \mathbb{N} \rightarrow \mathbb{N}$ such that each graph of clique-width $\text{cw}$ can be clique colored with at most $g(\text{cw})$ many colors?	solved	Yes, with $g(k) \in \mathcal{O}(k)$ . (G)
10.5	Is there an $n^{2^{2^{o(\text{cw})}}}$ -time algorithm for CLIQUE COLORING, where $\text{cw}$ denotes the clique-width of the input graph, or would such an algorithm refute ETH?	open	
10.6	Is there a $2^{2^{2^{o(\text{cw})}}} \cdot n^{\mathcal{O}(1)}$ -time algorithm for 2-CLIQUE COLORING or would such an algorithm refute ETH?	open	
10.7	Is $b$ -COLORING NP-complete on circular arc graphs, or, more generally, on any graph class of constant mim-width?	solved	NP-c on $\bigcup$ circular arc
10.8	Is $b$ -COLORING parameterized by the mim-width of a given branch decomposition of the input graph plus the number of colors XP?	solved	Yes. (B)
10.9	For which function $f: \mathbb{N} \rightarrow \mathbb{N}$ does it hold that for all fixed $k \geq 3$ , $b$ -COLORING on graphs of clique-width $\text{cw}$ can be solved in time $\mathcal{O}^*(f(k)^{\text{cw}})$ while an algorithm running in time $\mathcal{O}^*((f(k) - \epsilon)^{\text{cw}})$ , for any $\epsilon > 0$ , would refute SETH? What about FALL COLORING, or CLIQUE COLORING?	open	
10.10 (I)	Is $b$ -COLORING parameterized by the treewidth of the input graph W[1]-hard?	solved	Yes. (D)
10.10 (II)	What is the fastest algorithm for $b$ -COLORING parameterized by treewidth under the ETH?	open	

(A). J., Lima, Lokshtanov [STACS 2021].

- (B). Bergougnoux, Dreier, J. [arXiv:2202.13335, SODA 2023].
- (C). Bakkane, J. [IPEC 2022]: Dichotomies for minimization problems, and for maximization problems when  $\sigma$  and  $\rho$  are finite.
- (D). J., Lima, Sharma [arXiv:2209.07772, ISAAC 2023]:  $b$ -COLORING parameterized by pathwidth is XNLP-complete, which implies it is  $W[t]$ -hard for all  $t$ .
- (E). Korhonen and Lokshtanov [arXiv:2211.07154, STOC 2023]: A  $2^{\mathcal{O}(k^2)}n^4$  time algorithm that given a graph  $G$  decides if  $G$  has a tree decomposition of width at most  $k$ .
- (F). Bergougnoux, Hamm, J., Lima [ESA 2025]: INDEPENDENT SET and DOMINATING SET given a linear branch decomposition of mim-width  $w$  cannot be solved in  $n^{o(w)}$  time unless the ETH fails.
- (G). O. Kwon observed that one step the proof of the  $\chi$ -boundedness of graphs of bounded rank-width by Dvořák and Král' [Eur. J. Combin. 2012] shows that every graph with a decomposition of module-width  $k$  has a  $3k$ -clique coloring. Since clique-width is linearly upper bounded by module-width, the claim follows.
- (H). Computing mim-width  $k$  exactly is NP-hard for fixed  $k$ , as shown by Bergougnoux, Bonnet, and Duron [arXiv:2501.05638, ICALP 2025].